

## Cube Root of Perfect Cubes

Cube Roots can be calculated mainly through inspection (Vilokanam) and argumentation.

### First Principles

To be able to perform cube roots, one must be familiar with the cubes and of the first nine natural numbers. Thus, including the cube of zero, we have:

- Cube of 1 is  $0^3 = 0$
- Cube of 1 is  $1^3 = 1$
- Cube of 2 is  $2^3 = 8$
- Cube of 3 is  $3^3 = 27$
- Cube of 4 is  $4^3 = 64$
- Cube of 5 is  $5^3 = 125$
- Cube of 6 is  $6^3 = 216$
- Cube of 7 is  $7^3 = 343$
- Cube of 8 is  $8^3 = 512$
- Cube of 9 is  $9^3 = 729$

We observe that the cubes of zero and the first nine natural numbers have their own distinctive endings

- cubes of 0, 1, 4, 5, 6 and 9 end with the same digit
- cubes of 2 & 8 and 3 & 7, complements from 10, end with their ten's complement. In other words, the last digit being 8 & 2 and 7 and 3 respectively.

*These observations will help us to find out the last digit in the cube root of a perfect cube merely by inspection, Vilokanam.*

As we can see, the cubes of the first nine natural numbers have a minimum of one and a maximum of three digits.

***The number of digits in the cube root is the same as the number of 3-digit groups in the original cube.***

Double or single digits, which may remain at the left, are taken as occupying one single 3-digit group all by themselves.

So, for instance a perfect cube that contains, 4, 5 or 6 digits will have two 3-digit groups and so the cube root will have **two digits**.

Or, a perfect cube that contains, 7, 8 or 9 digits will have three 3-digit groups and so the cube root will have **three digits**.

Yet again, a perfect cube that contains, 10, 11 or 12 digits will have four 3-digit groups and so the cube root will have **four digits**.

The first digit of the cube root can be seen by inspection, from the first group in the cube.

So, the number of digits, the first and last digits of the cube root of a perfect cube.

Here are a few examples

**Example 1:**  $\sqrt[3]{2197}$ 

First of all we form groups from the right and so in this example there will only be two groups: 197 and 2. So this perfect cube has two digits in the cube root.

Next we observe the left-most group and see what is the largest cube that can be subtracted from it. We see that the largest cube to be subtracted is 1 and so this forms the left-most digit of the cube root.

Next we see that of the right-most group of three digits, 197, the last digit (unit's place digit) is a 7.  $3^3$  gives 27 (**ending in 7!**) so we conclude that the last digit in cube root must be 3.

Therefore cube root must be 13.

$$\begin{array}{r} \text{eg(1)} \quad \sqrt[3]{2197} \\ = \quad \begin{array}{r} 2 \quad 197 \\ 1 \quad 3 \end{array} \\ \hline 13 \end{array}$$

**Example 2:**  $\sqrt[3]{314432}$ 

Once again, we form groups from the right and so in this example there will again only be two groups: 432 and 314. So this perfect cube has two digits in the cube root.

Next we observe the left-most group as see what is the largest cube that can be subtracted from it. We see that the largest cube to be subtracted is 6, as  $6^3$  is 216 and  $17^3$  is 343 (too much!) and so this forms the left-most digit of the cube root.

Next we see that of the right-most group of digits, 432, the last digit (unit's place digit) is a 2.  $8^3$  gives 512 (**ending in 2!**) so we conclude that the last digit in cube root must be 8.

Therefore the cube root must be 68.

$$\begin{array}{r} \text{eg(2)} \quad \sqrt[3]{314432} \\ = \quad \begin{array}{r} 314 \quad 432 \\ 6 \quad 8 \end{array} \\ \hline 68 \end{array}$$

So, just by Vilokanam, mere observation, the cube root of perfect cubes of between four to six digits can easily be worked out and will be a number between 10 and 99.

Example	Cube	Digits of Cube Root		Cube Root
		Left	Right	
3	19683	2	7	27
4	35937	3	3	33
5	97336	4	6	46
6	195112	5	8	58
7	262144	6	4	64
8	493039	7	9	79
9	531441	8	1	81
10	970299	9	9	99

The cube root of a perfect cube having 7 to 9 digits will be a three digit number as the number of groups formed will be three. From the right, there will be a group of three digits; in the middle there will be another group of three; and the left-most group will be composed of 1, 2 or 3 digits.

In this case, to find the left-most digit and the last digit from right-most group we can use Vilokanam....inspection. Working out the middle digit of the cube root involves a little more effort.

### The algebra behind cubing a three digit number (abc)

Suppose that we wish find the cube of a three digit number **abc**. We can write the cube of this number algebraically as follows:

$$(ax^2+bx+c)^3=(ax^2+bx+c)(a^2x^4+b^2x^2+c^2+2abx^3+2acx^2+2bcx)$$

$$\begin{aligned}(ax^2+bx+c)^3 &= a^3x^6+2a^2bx^5+a^2b^2x^4+2a^2cx^4+2abcx^3+a^2c^2x^2 \\ &\quad +a^2bx^5+2ab^2x^4+b^3x^3+2abcx^3+2b^2cx^2+bc^2x \\ &\quad +a^2cx^4+2abcx^3+b^2c^2x^2+2ac^2x^2+2bc^2x+c^3\end{aligned}$$

Grouping like terms in orders of x:

$$\begin{aligned}(ax^2+bx+c)^3 &= a^3x^6+3a^2bx^5+(3ab^2+3a^2c)x^4+(6abc+b^3)x^3 \\ &\quad +(3ac^2+3b^2c)x^2+3bc^2x+c^3\end{aligned}$$

In Tabular form (*Using Base 10 i.e x = 10*)

M	H Th	T Th	Th	H	T	U
$x^6$	$x^5$	$x^4$	$x^3$	$x^2$	$x^1$	$x^0$
$a^3$	$3a^2b$	$3ab^2+3a^2c$	$6abc+b^3$	$3ac^2+3b^2c$	$3bc^2$	$c^3$

So, here we can clearly see the coefficients of the various powers of x.

1. The units' place is determined by  $c^3$
2. The tens' place is determined by  $3bc^2$
3. The hundreds place is determined by  $3ac^2 + 3b^2c$
4. The thousands place is determined by  $6abc + b^3$
5. The ten thousands place is determined by  $3ab^2 + 3a^2c$
6. The hundred thousands place is determined by  $3a^2b$
7. The millions place is determined by  $a^3$

So, how can this help us determine the cube root of a perfect cube?

By means of sorting the various parts of the algebraic expression into their respective place we can eliminate unknown coefficients, a, b c etc. The method used is of an argumentational nature. Here goes....

#### Method of Finding the Cube Root of a Perfect Cube

1. From the units place, subtract  $c^3$ . This eliminates the last digit
2. From the tens place subtract  $3bc^2$  thus eliminating the penultimate digit
3. From the hundred's place subtract  $3ac^2 + 3b^2c$  eliminating pre-penultimate digit
4. From the thousand's place subtract  $6abc + b^3$  eliminating the pre-pre-penultimate digit and so on....

For perfect cubes we have the added advantage of knowing the last (unit's place) digit.

Now for a working example.

**Example 11:** What is  $\sqrt[3]{109902239}$  ?

1. First of all, partitioning the perfect cube
 

109	902	239
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2. By inspection, the last digit in the right-most group of digits, 239, is 9. This implies that c must be 9., because  $9^3 = 729$  as this is the cube which ends in 9. In other words **c = 9**
3. Inspecting the left-most group of digits, 109, we see that the largest cube less than 109 is 64 implying that the first digit of the cube root is 4, i.e. **a = 4**.
4. So, now we know that the cube root consists of the digits: 4 ? 9. All we need to do is to obtain the middle digit. So, we next subtract  $c^3$  from the perfect cube. ( **$c^3 = 729$** )
 

109	902	239
		-729
109	901	510
5. The penultimate digit can now be seen and in this example it is 1. Now, according to our algebraic analysis the penultimate term is constructed using  $3bc^2$ .

So:

$3 \times b \times 9^2$  ends in 1. Or, in other words,  $243 \times b$  ends in 1.

But multiplying  $243 \times 7 = 1701$  and this number ends in 1 so we conclude that b must be 7.

Voila we have found the middle number, **b = 7**

So, we have  $\sqrt[3]{109902239} = 479$

**Example 12:** What is  $\sqrt[3]{259694072}$  ?

1. First of all, partitioning the perfect cube 

259	694	072
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2. By inspection, the last digit in the right-most group of digits, 072, is 8. This implies that  $c$  must be 8, because  $8^3 = 512$  as this is the cube which ends in 2. In other words  $c = 8$
3. Inspecting the left-most group of digits, 259, we see that the largest cube less than 259 is 216 implying that the first digit of the cube root is 6, i.e.  $a = 6$ .
4. So, now we know that the cube root consists of the digits: 6 ? 8. All we need to do is to obtain the middle digit. So, we next subtract  $c^3$  from the perfect cube. ( $c^3 = 512$ ) 

259	694	072
		-512
259	643	560
5. The penultimate digit can now be seen and in this example it is 6. Now, according to our algebraic analysis the penultimate term is constructed using  $3bc^2$ .
6. So:  $3 \times b \times 8^2$  ends in 6. Or, in other words,  $192 \times b$  ends in 6. But multiplying  $192 \times 3 = 576$  and this number ends in 6 so we conclude that  $b$  must be 3.  
Voila we have found the middle number,  $b = 3$

So we have  $\sqrt[3]{259694072} = 638$

**Example 13:** What is  $\sqrt[3]{994076141}$  ?

1. First of all, partitioning the perfect cube 

994	76	141
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2. By inspection, the last digit in the right-most group of digits, 141, is 1. This implies that  $c$  must be 1, because  $1^3 = 1$  as this is the cube which ends in 1. In other words  $c = 1$
3. Inspecting the left-most group of digits, 994, we see that the largest cube less than 994 is 729 implying that the first digit of the cube root is 9, i.e.  $a = 9$ .
4. So, now we know that the cube root consists of the digits: 9 ? 1. All we need to do is to obtain the middle digit. So, we next subtract  $c^3$  from the perfect cube. ( $c^3 = 1$ ) 

994	076	141
		-1
994	076	140
5. The penultimate digit can now be seen and in this example it is 4. Now, according to our algebraic analysis the penultimate term is constructed using  $3bc^2$ .
6. So:  $3 \times b \times 1^2$  ends in 4. Or, in other words,  $3 \times b$  ends in 4. But multiplying  $3 \times 8 = 24$  and this number ends in 4 so we conclude that  $b$  must be 8.  
Voila we have found the middle number,  $b = 8$

So we have  $\sqrt[3]{994076141} = \mathbf{981}$